Final Exam 2013 Winter Term 2 Solutions

1. (a) Let $Q : -x + 5y - 3z = 2$ and $R : 3x - \frac{9}{5}y - 4z = 0$ be two planes. Determine if Q and R are parallel, orthogonal, or identical.

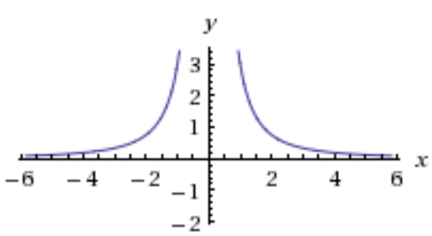
Solution: First, we identify the normal vectors of each plane: $\mathbf{n}_Q = \langle -1, 5, -3\rangle$ and $\mathbf{n}_R = \langle 3, -\frac{9}{5}, -4\rangle$. We first check if they are parallel. If they are, then so are their normal vectors, and $\mathbf{n}_Q = c\mathbf{n}_R$ for some constant c, that is $\langle -1, 5, -3\rangle = \langle 3c, -\frac{9}{5}c, -4c\rangle$. Then, $-1 = 3c$ gives $c = -\frac{1}{3}$, whereas $-3 = -4c$ gives $c = \frac{3}{4}$. So, no such c exists and the planes are not parallel. Note that if the planes are identical, then in particular, they have to be parallel, so this also means that they are not identical. Last, we check for orthogonality.

$$-1(3) + 5\left(-\frac{9}{5}\right) + (-3)(-4) = -3 - 9 + 12 = 0.$$

So, Q and R are orthogonal to each other.

(b) The volume of a right circular cone of radius x and height y is $V(x, y) = \dfrac{\pi x^2 y}{3}$. Graph the level curve $V(x, y) = \pi$.

Solution: The equation for the level curve $V(x, y) = \pi$ is $\pi = \frac{\pi x^2 y}{3}$, or equivalently, $3 = x^2 y$. The graph is:



(c) Let $f(x, y) = \sin(xy)$. Find $\dfrac{\partial^2 f}{\partial x \partial y}$.

Solution: We get:

$$\frac{\partial f}{\partial y} = x\cos(xy),$$
$$\frac{\partial^2 f}{\partial x \partial y} = \cos(xy) - xy\sin(xy).$$

(d) Use sigma notation to write the midpoint Riemann sum for $f(x) = x^8$ on $[5, 15]$ with $n = 50$. Do not evaluate the midpoint Riemann sum.

Solution: We have $a = 5, b = 15, n = 50$, so $\Delta x = \dfrac{b-a}{n} = 0.2$. For midpoint Riemann sums,

$$x_k^* = a + (k - 1/2)\Delta x = 5 + (k - 1/2)0.2 = 4.9 + 0.2k.$$

Then, the Midpoint Riemann sum for $f(x) = x^8$ on $[5, 15]$ with $n = 50$ using sigma notation is:

$$\sum_{k=1}^{50} f(x_k^*)\Delta x = \sum_{k=1}^{50} (4.9 + 0.2k)^8 (0.2).$$

(e) Evaluate $\displaystyle\int_1^5 f(x)\,dx$, where $f(x) = \begin{cases} 3 & \text{if } x \leq 3, \\ x & \text{if } x \geq 3. \end{cases}$.

Solution: We have that:

$$\int_1^5 f(x)\,dx = \int_1^3 3\,dx + \int_3^5 x\,dx = 3x\,\Big|_1^3 + \frac{x^2}{2}\,\Big|_3^5 = (9-3) + \left(\frac{25}{2} - \frac{9}{2}\right) = 6 + 8 = 14.$$

(f) If $f'(1) = 2$ and $f'(2) = 3$, find $\displaystyle\int_1^2 f'(x)f''(x)\,dx$.

Solution: Using direct substitution with $u = f'(x)$ and $du = f''(x)dx$, we get:

$$\int_1^2 f'(x)f''(x)\,dx = \int_2^3 u\,du = \frac{u^2}{2}\,\Big|_2^3 = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}.$$

(g) Evaluate $\int \arccos(y)\,dy$.

Solution: Using integration by parts with $u = \arccos(y)$, $du = -\frac{1}{\sqrt{1-y^2}}\,dy$, and $dv = dy$, $v = y$, we get:

$$\int \arccos(y)\,dy = y\arccos(y) + \int \frac{y}{\sqrt{1-y^2}}\,dy.$$

For the remaining integral, using a direct substitution with $x = 1 - y^2$, and $dx = -2ydy$, we get:

$$\int \frac{y}{\sqrt{1-y^2}}\,dy = -\frac{1}{2}\int x^{-1/2}\,dx = -x^{1/2} + C = -\sqrt{1-y^2} + C.$$

Thus,

$$\int \arccos(y)\, dy = y \arccos(y) - \sqrt{1-y^2} + C.$$

(h) Evaluate $\int \cos^3(x)\sin^4(x)\, dx$.

Solution: Using direct substitution with $u = \sin(x)$, $du = \cos(x)\, dx$, and the identity $\cos^2(x) + \sin^2(x) = 1$, we get:

$$\begin{aligned}\int \cos^3(x)\sin^4(x)\, dx &= \int \cos^2(x)\sin^4(x)\cos(x)\, dx \\ &= \int (1-\sin^2(x))\sin^4(x)\cos(x)\, dx = \int (1-u^2)u^4\, du \\ &= \int (u^4 - u^6)\, du = \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C.\end{aligned}$$

(i) Evaluate $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$.

Solution: Completing the square, we get $3 - 2x - x^2 = 4 - (x+1)^2$, so first, let $u = x+1$ and $du = dx$.

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{du}{\sqrt{4-u^2}}.$$

Using trigonometric substitution with $u = 2\sin\theta$ (so $\theta = \arcsin\left(\frac{u}{2}\right)$), $du = 2\cos\theta d\theta$, we get:

$$\int \frac{du}{\sqrt{4-u^2}} = \int \frac{2\cos\theta d\theta}{2\cos\theta} = \int d\theta = \theta + C$$

$$\Rightarrow \int \frac{dx}{\sqrt{3-2x-x^2}} = \arcsin\left(\frac{x+1}{2}\right) + C.$$

(j) Evaluate $\displaystyle\int \frac{x-13}{x^2-x-6}\, dx$.

Solution: Since $x^2 - x - 6 = (x-3)(x+2)$, we may use partial fraction, and let:

$$\frac{x-13}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)} = \frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}.$$

So, $A+B=1$ and $2A-3B=-13$. $A+B=1$ gives $B=1-A$, so $-13 = 2A-3B = 2A-3+3A = -3+5A$. So, $A=-2$ and $B=3$. Thus,

$$\int \frac{x-13}{x^2-x-6}\,dx = \int \left(\frac{-2}{x-3} + \frac{3}{x+2}\right) dx = -2\ln|x-3| + 3\ln|x+2| + C.$$

(k) The random variable X has probability density function

$$f(x) = \begin{cases} 0, & \text{if } x < 1, \\ \frac{3}{2}x^{-5/2}, & \text{if } x \geq 1. \end{cases}$$

Find the expected value $\mathbb{E}(X)$ of the random variable X.

Solution: The expected value $\mathbb{E}(X)$ is:

$$\begin{aligned}\int_{-\infty}^{\infty} xf(x)\,dx &= \int_1^{\infty} \frac{3}{2}x^{-3/2}\,dx \qquad \text{(since } f(x)=0 \text{ for } x<1) \\ &= \lim_{b\to\infty}\int_1^b \frac{3}{2}x^{-3/2}\,dx = \lim_{b\to\infty} -3x^{-1/2}\,|_1^b = \lim_{b\to\infty} -3b^{-1/2}+3 \\ &= 3.\end{aligned}$$

(l) Evaluate $\displaystyle\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n + \left(-\frac{2}{5}\right)^{n-1}\right]$.

Solution: We have that:

$$\begin{aligned}\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n + \left(-\frac{2}{5}\right)^{n-1}\right] &= \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(-\frac{2}{5}\right)^{n-1} \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{1}{1-(-\frac{2}{5})} \\ &= \frac{1}{2} + \frac{5}{7} = \frac{17}{14}.\end{aligned}$$

> Remark: *We can split the sum in the first line into two because we recognize that both series on the right hand side are convergent geometric series because the ratios are between −1 and 1. This is important, yet subtle, because such splitting is not possible if we do not know the convergence of these series.*

(m) Let k be a constant. Find the value of k such that $f(x) = 1 + k|x|$ is a probability density function on $-1 \le k \le 1$.

Solution: First, using definition of absolute values, we can write $f(x)$ in a more explicit form as follows:

$$f(x) = \begin{cases} 1 - kx, & -1 \le x < 0, \\ 1 + kx, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

In order to be a probability density function, we need $\int_{-\infty}^{\infty} f(x)\,dx = 1$, so:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx \\ &= \int_{-1}^{0} (1 - kx)\,dx + \int_{0}^{1} (1 + kx)\,dx \qquad (\text{ since } f(x) = 0 \text{ outside of this interval }) \\ &= (x - \frac{kx^2}{2}) \mid_{-1}^{0} + (x + \frac{kx^2}{2}) \mid_{0}^{1} = 1 + k/2 + 1 + k/2 = 2 + k. \end{aligned}$$

So, $1 = 2 + k$ which gives $k = -1$.

(n) Let $h(s)$ be a continuous function with $h(10) = 2$. If $f(x, y) = \int_{1}^{xy} h(s)\,ds$, find $f_x(2, 5)$.

Solution: By the Fundamental Theorem of Calculus Part 1 and chain rule, we get:

$$\begin{aligned} & f_x(x, y) = yh(xy) \\ \Rightarrow\ & f_x(2, 5) = 5h(10) = 10. \end{aligned}$$

2. (a) Determine whether the series $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}}$ converges.

Solution: We want to use the Limit Comparison Test with the second series being $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4}}{\sqrt{k^5}}$. We have that:

$$\lim_{k\to\infty} \frac{a_n}{b_n} = \lim_{k\to\infty} \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \cdot \frac{\sqrt{k^5}}{\sqrt[3]{k^4}} = \lim_{k\to\infty} \frac{\sqrt[3]{1+\frac{1}{k^4}}}{\sqrt{1+\frac{9}{k^5}}} = 1.$$

So, either both series converge or both diverge. Note that $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4}}{\sqrt{k^5}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{7}{6}}}$, which is a p-series with $p = \frac{7}{6} > 1$. Hence, this series converges by the p-series Test, and therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}}$ also converges.

(b) Find the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{10^{k+1}(k+1)!}$.

Solution: Using the Ratio Test, we get:

$$\begin{aligned} L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{x^{k+1}}{10^{k+2}(k+2)!} \cdot \frac{10^{k+1}(k+1)!}{x^k}\right| \\ &= \lim_{k\to\infty} \left|\frac{x}{10(k+2)}\right| = 0. \end{aligned}$$

Note that the series converges for $L < 1$, so since $L = 1$, the series always converges for any x. Thus, the radius of convergence is ∞.

(c) Let $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ be the Maclaurin series for $f(x) = \dfrac{3}{x+1} - \dfrac{1}{2x-1}$, i.e., $\displaystyle\sum_{n=0}^{\infty} b_n x^n = \frac{3}{x+1} - \frac{1}{2x-1}$. Find b_n.

Solution: We have that:

$$\begin{aligned}\frac{3}{x+1}-\frac{1}{2x-1}&=3\left(\frac{1}{1-(-x)}\right)+\frac{1}{1-2x}\\&=3\sum_{n=0}^{\infty}(-x)^n+\sum_{n=0}^{\infty}(2x)^n\\&=\sum_{n=0}^{\infty}3(-1)^n x^n+\sum_{n=0}^{\infty}2^n x^n=\sum_{n=0}^{\infty}(3(-1)^n+2^n)x^n.\end{aligned}$$

Hence, $b_n = 3(-1)^n + 2^n$.

3. (a) Use the method of Lagrange multipliers to find the maximum and minimum values of $(x+1)^2 + (y-2)^2$ on the circle $x^2 + y^2 = 125$.

Solution: We want to find the extrema of $f(x,y) = (x+1)^2 + (y-2)^2$ subject to the constraint $g(x,y) = x^2 + y^2 - 125 = 0$. So, by Lagrange multipliers, we want to solve the following system of equations:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x+2 = \lambda 2x \\ 2y-4 = \lambda 2y \\ x^2+y^2-125 = 0 \end{cases}$$

Note that $\lambda \neq 1$ (otherwise the first two equations do not make sense). So, we get $x = \frac{1}{\lambda-1}$, and $y = \frac{2}{1-\lambda}$. So,

$$\frac{1}{(\lambda-1)^2}+\frac{4}{(1-\lambda)^2}-125=0 \Rightarrow \frac{5}{(1-\lambda)^2}=125 \Rightarrow \frac{1}{25}=(1-\lambda)^2 \Rightarrow \lambda = 1 \pm \frac{1}{5}.$$

If $\lambda = \frac{4}{5}$, then $x = -5$, $y = 10$ and $f(-5,10) = 100$. If $\lambda = \frac{6}{5}$, then $x = 5$, $y = -10$ and $f(5,-10) = 180$. So, subject to the constraint $g(x,y) = 0$, the maximum value of $f(x,y)$ is 180 and the minimum value is 100.

(b) Find the point on the circle $x^2 + y^2 = 125$ that has minimum distance from the point $(-1,2)$.

Solution: The distance from a point (x,y) to the point $(-1,2)$ is given by: $\sqrt{(x+1)^2+(y-2)^2}$. Since square root is an increasing function, minimizing $\sqrt{(x+1)^2+(y-2)^2}$ is equivalent to minimizing $(x+1)^2+(y-2)^2$. So, we are looking for the point (x,y) where $f(x,y) = (x+1)^2 + (y-2)^2$ attains its minimum value subject to the constraint $x^2 + y^2 = 125$. By part (a), the minimum distance is attained at the point $(-5, 10)$.

4. Let $T(x, y) = \frac{1}{9}y^3 + x^2 - 2xy + 6x - 6y$.

(a) Find all critical points of $T(x, y)$ and classify each as a local maximum, local minimum, or saddle points.

Solution: First, note that the domain of T is all of $\mathbb{R}^2$ since it is a polynomial, so the only critical points are those where both partial derivatives are zero. Computing the first partial derivatives, we get:

$$T_x(x, y) = 2x - 2y + 6, \qquad T_y(x, y) = \frac{1}{3}y^2 - 2x - 6.$$

$T_x(x, y) = 0$ gives $x = y - 3$. So, $0 = T_y(x, y) = \frac{1}{3}y^2 - 2x - 6 = \frac{1}{3}y^2 - 2y = y(\frac{1}{3}y - 2)$. We get $y = 0$ (so $x = -3$) or $y = 6$ (so $x = 3$). Thus, we get two critical points $(-3, 0)$ and $(3, 6)$. To classify those points, we need the second-order partial derivatives and the discriminant:

$$f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = \frac{2}{3}y, \quad f_{xy}(x, y) = -2, \quad D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{3}y - 4.$$

Using the Second Derivative Test, we can classify the points as follows:

- At $(-3, 0)$, $D(-3, 0) = -4 < 0$, so $(-3, 0)$ is a saddle point.
- At $(3, 6)$, $D(3, 6) = 4 > 0$, and $f_{xx}(3, 6) = 2 > 0$, so $(3, 6)$ is a local minimum.

(b) Find the maximum and minimum values of $T(x, y)$ on the region

$$R = \left\{(x, y) : \frac{1}{3}y^2 - 3 \leq x \leq 0\right\}.$$

Solution: From part (a), $T(x, y)$ has two critical points $(-3, 0)$ and $(3, 6)$. We need to test to see if they lie in R:

- For $(-3, 0)$, we do have $-3 \leq 0$, and $\frac{1}{3}(0^2) - 3 \leq -3$, so $(-3, 0)$ is in R.
- For $(3, 6)$, the x-value $3 > 0$, so $(3, 6)$ is not in R.

On the boundary of R which has two parts: the vertical segment and the parabolic arc. Note that end points are $(0, 3)$ and $(0, -3)$. We have that:

- On the vertical segment $x = 0$ (for $-3 \le y \le 3$): $h_1(y) = T(0, y) = \frac{1}{9}y^3 - 6y$ has derivative $h_1'(y) = \frac{1}{3}y^2 - 6 = 0$ for $y = \pm 3\sqrt{2}$, neither of which lies in the domain $-3 \le y \le 3$. So, we get no candidates from this segment.

- On the parabolic arc $x = \frac{1}{3}y^2 - 3$ (for $-3 \le y \le 3$): we get:

$$\begin{aligned} h_2(y) &= \frac{1}{9}y^3 + \left(\frac{1}{3}y^2 - 3\right)^2 - 2\left(\frac{1}{3}y^2 - 3\right)y + 6\left(\frac{1}{3}y^2 - 3\right) - 6y \\ &= \frac{1}{9}y^3 + \frac{1}{9}y^4 - 2y^2 + 9 - \frac{2}{3}y^3 + 6y + 2y^2 - 18 - 6y \\ &= \frac{1}{9}y^4 - \frac{5}{9}y^3 - 9. \end{aligned}$$

So, $h_2'(y) = \frac{4}{9}y^3 - \frac{5}{3}y^2 = y^2\left(\frac{4}{9}y - \frac{5}{3}\right) = 0$ for $y = 0$ and $y = \frac{15}{4} > 3$ (not in the domain!). We get one more candidate: $(-3, 0)$.

Computing the values of $T(x, y)$ at those candidates:

$$T(-3, 0) = -9, \quad T(0, 3) = -15, \quad T(0, -3) = 15.$$

So, the maximum value of $T(x, y)$ on R is 15, and the minimum value of $T(x, y)$ on R is -15.

5. (a) Solve the initial value problem:

$$B'(t) = aB - m \text{ for } t \ge 0$$

, with $a = 0.02$, and $B(0) = B_0 = \$30,000$.

Solution: We have:

$$\frac{dB}{dt} = aB - m \Rightarrow \frac{dB}{aB - m} = dt.$$

Integrating each side separately, we get:

$$\int \frac{dB}{aB - m} = \int dt \Rightarrow \frac{1}{a}\ln|aB - m| = t + C.$$

Using $a = 0.02$, and $B(0) = B_0 = \$30,000$, we can solve for C:

$$\frac{1}{0.02}\ln|0.02(30000) - m| = 0 + C \Rightarrow C = 50\ln|600 - m|.$$

Thus, the solution to the initial value problem is:

$$50\ln|0.02B - m| = t + 50\ln|600 - m|.$$

(b) If $a = 0.02$ and $B(0) = B_0 = \$30,000$, what is the annual withdrawal rate m that ensure a constant balance in the account?

Solution: Constant balance means that the rate of change of B over time is 0, that is, $\frac{dB}{dt} = 0$. Furthermore, since $B_0 = \$30,000$, it means $B(t) = 30,000$ for all t. Then,

$$0 = 0.02(30000) - m = 600 - m \Rightarrow m = 600.$$

So, $m = 600$ gives a constant balance.

6. (a) Evaluate the sum of the convergent series:

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!}.$$

Solution: Note that we have:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ for any } x.$$

So,

$$\sum_{k=0}^{\infty} \frac{1}{\pi^k k!} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{\pi}\right)^k}{k!} = e^{\left(\frac{1}{\pi}\right)}.$$

Note that the series in the question starts at $k = 1$, not at $k = 0$, so:

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = \sum_{k=0}^{\infty} \frac{1}{\pi^k k!} - 1 = e^{\left(\frac{1}{\pi}\right)} - 1.$$

(b) Assume that the series $\displaystyle\sum_{n=1}^{\infty} \frac{na_n - 2n + 1}{n+1}$ converges, where $a_n > 0$ for $n = 1, 2, \ldots$. Is the following series

$$-\ln(a_1) + \sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right)$$

convergent? If your answer is NO, justify your answer. If your answer is YES, evaluate the sum of the series $-\ln(a_1) + \displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right)$.

Solution: Consider the k-th partial sum of the series $-\ln(a_1)+\sum_{n=1}^{\infty}\ln\left(\frac{a_n}{a_{n+1}}\right)$, which is defined by:

$$s_k = -\ln(a_1)+\sum_{n=1}^{k}\ln\left(\frac{a_n}{a_{n+1}}\right) = -\ln(a_1)+\sum_{n=1}^{k}(\ln(a_n)-\ln(a_{n+1})) = -\ln(a_{k+1}).$$

By definition, the series $-\ln(a_1)+\sum_{n=1}^{\infty}\ln\left(\frac{a_n}{a_{n+1}}\right)$ converges to a value L if and only if the sequence of partial sums $\{s_k\}$ converges to the same value L. Thus, we want to evaluate the limit $\lim_{k\to\infty} -\ln(a_{k+1})$ if it exists. Since $\sum_{n=1}^{\infty}\frac{na_n-2n+1}{n+1}$ converges, by the Divergence Test,

$$0=\lim_{n\to\infty}\frac{na_n-2n+1}{n+1}=\lim_{n\to\infty}\frac{a_n-2+\frac{1}{n}}{1+\frac{1}{n}}=\left(\lim_{n\to\infty}a_n\right)-2.$$

So, $\lim_{n\to\infty}a_n=2$. Therefore,

$$\lim_{k\to\infty}-\ln(a_{k+1})=-\ln(2).$$

Hence, the series $-\ln(a_1)+\sum_{n=1}^{\infty}\ln\left(\frac{a_n}{a_{n+1}}\right)$ converges to $-\ln(2)$.